Chapter 1: Relations and Functions

Relations

Let's start by understanding how we can define a relationship between elements of sets.

🧠 **Memory Boost: Class 11 Prerequisite Check**

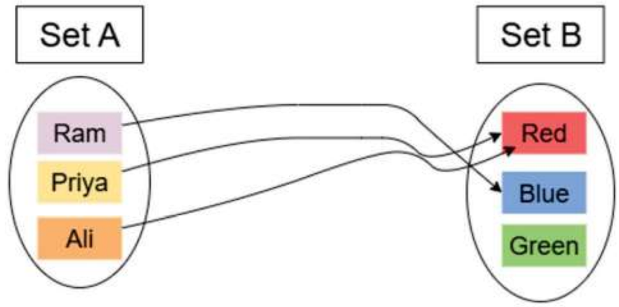
Before we dive into Relations, let's refresh two key ideas from Class 11: **Sets** and **Cartesian Product**.

- **Set**: A set is a well-defined collection of distinct objects. Example: $A = \{1, 2, 3\}$.
- **Cartesian Product**: The Cartesian product of two non-empty sets, A and B, denoted by $A \times B$, is the set of all possible **ordered pairs** where the first element is from A and the second element is from B.
  - **Formula:** $A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$.
  - **Example**: If $A = \{1, 2\}$ and $B = \{x, y\}$, then $A \times B = \{(1, x), (1, y), (2, x), (2, y)\}$. Remember, the order matters, so $(1, x)$ is not the same as $(x, 1)$.

📚 A **Relation** R from a non-empty set A to a non-empty set B is simply a **subset** of the Cartesian product $A \times B$. We define this subset by describing a relationship between the first element and the second element of the ordered pairs.

- If we define a relation on a single set A, it is a subset of $A \times A$.
- **Real-World Analogy**: Think of a "students and their house colors" list in a school. Let Set A be students {Ram, Priya, Ali} and Set B be house colors {Red, Blue, Green}. A relation R

could be "belongs to house," resulting in ordered pairs like {(Ram, Blue), (Priya, Red), (Ali, Red)}. This set of pairs is a subset of the full Cartesian product $A \times B$.



- **Domain, Codomain, and Range of a Relation**:
  - **Domain**: The set of all first elements of the ordered pairs in a relation R. In our example, the domain is {Ram, Priya, Ali}.
  - **Codomain**: The entire set B. In our example, the codomain is {Red, Blue, Green}.
  - **Range**: The set of all second elements of the ordered pairs in a relation R. The range is always a subset of the codomain. In our example, the range is {Red, Blue}.

Types of Relations

Let R be a relation on a set A.

Type of Relation	Condition	Simple Explanation
Reflexive	For every $a \in A$, $(a, a) \in R$.	Every element must be related to itself.
Symmetric	If $(a, b) \in R$, then $(b, a) \in R$.	If 'a' is related to 'b', then 'b' must be related to 'a'.
Transitive	If $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$.	If 'a' is related to 'b' and 'b' is related to 'c', then 'a' must be related to 'c'.

Equivalence Relation: A relation that is **reflexive, symmetric, and transitive** is called an equivalence relation. This is a very important concept for your board exams.

⚡ Pro-Tip / Shortcut Box

- **To disprove a property**: The fastest way is to find a single counter-example.

- Not Reflexive? Find one element $a \in A$ such that $(a, a) \notin R$.
- Not Symmetric? Find one pair $(a, b) \in R$ such that $(b, a) \notin R$.
- Not Transitive? Find two pairs $(a, b) \in R$ and $(b, c) \in R$ such that $(a, c) \notin R$.

- **Trivial Relations**: The **empty relation** (ϕ) and the **universal relation** ($A \times A$) are called trivial relations. The universal relation is always an equivalence relation.

✍️ **Worked Example**

Question: Let L be the set of all lines in a plane and R be the relation in L defined as $R = \{(L_1, L_2): L_1 \text{ is parallel to } L_2\}$. Show that R is an equivalence relation.

Solution: Let L be the set of all lines in a plane and R be the relation in L defined by $R = \{(L_1, L_2): L_1 \parallel L_2\}$.

1. **Reflexivity**:
    - We know that any line L_1 is parallel to itself ($L_1 \parallel L_1$).
    - Therefore, $(L_1, L_1) \in R$ for all $L_1 \in L$.
    - Hence, R is **reflexive**.
2. **Symmetry**:
    - Let $(L_1, L_2) \in R$. This means L_1 is parallel to L_2 ($L_1 \parallel L_2$).
    - If $L_1 \parallel L_2$, then L_2 is also parallel to L_1 ($L_2 \parallel L_1$).
    - This implies $(L_2, L_1) \in R$.
    - Hence, R is **symmetric**.
3. **Transitivity**:
    - Let $(L_1, L_2) \in R$ and $(L_2, L_3) \in R$.
    - This means $L_1 \parallel L_2$ and $L_2 \parallel L_3$.
    - We know that lines parallel to the same line are parallel to each other. Thus, $L_1 \parallel L_3$.
    - This implies $(L_1, L_3) \in R$.
    - Hence, R is **transitive**.

Since R is reflexive, symmetric, and transitive, it is an **equivalence relation**.

⚠️ **Pitfall Analysis**

- **Mistake**: Forgetting to check the transitive property for all possible pairs. Sometimes students only check one or two examples and generalize.
  - **Correction**: Always assume arbitrary elements (a, b) and (b, c) are in R and then mathematically prove that (a, c) must be in R. If you suspect it's not transitive, be systematic in finding a counter-example.
- **Mistake**: In a question with numbers, like a relation on $\{1, 2, 3\}$, forgetting to check reflexivity for *all* elements. If $(1, 1)$ and $(2, 2)$ are in R, but $(3, 3)$ is not, the relation is not reflexive.
- **Mistake**: Confusing "symmetric" and "transitive". Remember, symmetric is about reversing a single pair, while transitive is about chaining two pairs together.

Functions

Now, let's move on to a special type of relation called a function.

🧠 **Memory Boost: Class 11 Prerequisite Check**

- **Function**: A function f from a set A to a set B is a specific rule that assigns **every** element of set A to **one and only one** element of set B. We write it as $f: A \to B$.
  - **Key points**: Every element in the domain must have an image. No element in the domain can have more than one image.

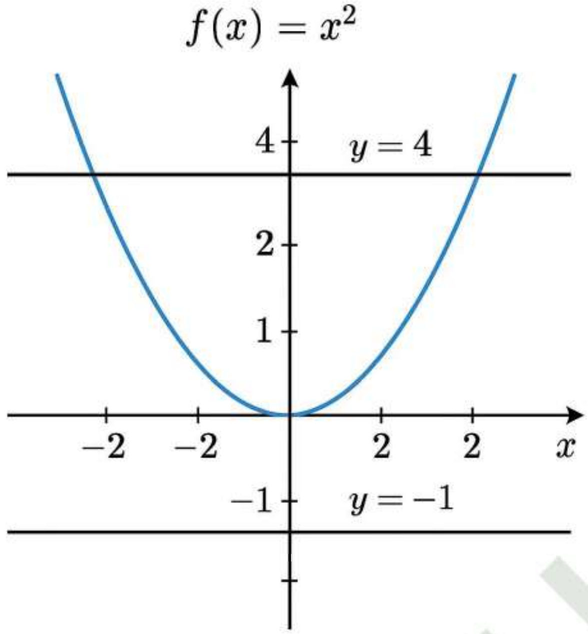
📚 In Class 12, we classify functions based on how elements are mapped.

Types of Functions (Mappings)

Type of Function	Condition	Simple Explanation
One-one (Injective)	For every $x_1, x_2 \in A$, if $f(x_1) = f(x_2)$, then $x_1 = x_2$.	Different inputs must have different outputs. No two inputs can map to the same output.
Many-one	A function that is not one-one.	At least two different inputs have the same output.
Onto (Surjective)	For every $y \in B$, there exists an $x \in A$ such that $f(x) = y$.	Every element in the codomain B must be the image of at least one element from the domain A. Nothing in the codomain is left out.
Into	A function that is not onto.	There is at least one element in the codomain that is not an image of any element from the domain.

Bijective Function: A function that is both **one-one and onto** is called a bijective function. Bijections are crucial for a function to be invertible.

- **Real-World Analogies**:
  - **One-one**: Assigning a unique Roll Number to each student in a class.
  - **Many-one**: Assigning a House (Red, Blue, Green) to students. Multiple students (inputs) can belong to the same house (output).
  - **Onto**: If a school has library cards for every single student, and every student has been issued a card, the function from cards to students is onto.



A graph of $f(x) = x^2$ is a parabola opening upwards. A horizontal line at $y = 4$ cuts the graph at $x = 2$ and $x = -2$. This shows it's a many-one function. A horizontal line at $y = -1$ doesn't cut the graph at all, showing that if the codomain is R, it's not an onto function.

Pro-Tip / Shortcut Box

- **Horizontal Line Test**: To quickly check if a function is one-one from its graph, draw horizontal lines. If *any* horizontal line intersects the graph more than once, the function is **many-one**. If every horizontal line intersects the graph at most once, it is **one-one**.
- **Proving One-one (Algebraic Method)**:
  1. Assume $f(x_1) = f(x_2)$ for two arbitrary elements x_1, x_2 in the domain.
  2. Algebraically manipulate the equation.
  3. If you can prove that this necessarily implies $x_1 = x_2$, the function is **one-one**.
- **Proving Onto (Algebraic Method)**:
  1. Let y be an arbitrary element in the codomain.
  2. Set $f(x) = y$.
  3. Solve for x in terms of y.

4. Check if this value of x is in the domain for every possible y in the codomain. If it is, the function is **onto**.

✍️ **Worked Example**

Question: Show that the function $f: R \to R$ defined by $f(x) = 2x + 3$ is a bijection.

Solution:

1. **To Prove One-one (Injectivity):**
   - Let $x_1, x_2 \in R$ (the domain).
   - Assume $f(x_1) = f(x_2)$.
   - $2x_1 + 3 = 2x_2 + 3$
   - $2x_1 = 2x_2$
   - $x_1 = x_2$
   - Since $f(x_1) = f(x_2) \implies x_1 = x_2$, the function is **one-one.**
2. **To Prove Onto (Surjectivity):**
   - Let $y \in R$ (the codomain).
   - Set $f(x) = y$.
   - $2x + 3 = y$
   - $2x = y - 3$
   - $x = \frac{y-3}{2}$
   - For any real number y, the value of $x = \frac{y-3}{2}$ will also be a real number.
   - Thus, for every y in the codomain, there exists an x in the domain such that $f(x) = y$.
   - Therefore, the function is **onto**.

Since $f(x)$ is both one-one and onto, it is a **bijection**.

⚠️ **Pitfall Analysis**

- **Mistake**: Incorrectly stating the Codomain and Range. The question will always define the codomain (e.g., $f: A \to B$). Your job is to check if the Range = Codomain for the 'onto' property. For $f(x) = x^2$ with $f: R \to R$, the range is $[0, \infty)$, which is not equal to the codomain R. So, it's not onto.

- **Mistake**: In the 'onto' proof, after finding x in terms of y, not verifying if that x is valid. For example, if $f: N \to N$ and you find $x = y - 5$, what happens if $y = 3$? Then $x = -2$, which is not in the domain N. So the function is not onto.

Composition of Functions and Invertible Functions

Here we learn how to combine functions and how to reverse them.

🧠 **Memory Boost: Quick Refresher**

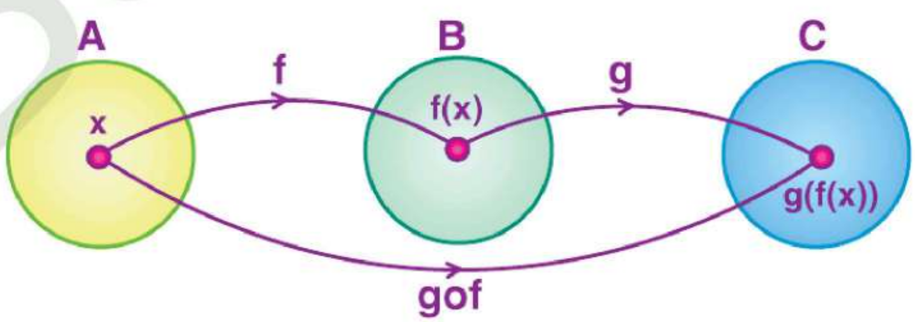
- **A function** $f: A \to B$ takes an input from set A and gives an output in set B.

📚 **Composition of Functions**

If we have two functions, $f: A \to B$ and $g: B \to C$, we can create a new function that goes directly from A to C. This is called the **composition** of g and f, denoted by $g \circ f$.

- **Definition:** $(g \circ f)(x) = g(f(x))$.

- **How it works**: First, apply the function f to x. Then, apply the function g to the result, $f(x)$.



Invertible Functions

A function $f: X \to Y$ is **invertible** if there exists another function $g: Y \to X$ such that $(g \circ f)(x) = x$ and $(f \circ g)(y) = y$. The function g is called the inverse of f and is denoted by f^{-1}.

- **The Golden Rule**: A function is invertible **if and only if** it is a **bijection** (both one-one and onto). This is a critical theorem.

⚡ Pro-Tip / Shortcut Box

- **Order Matters:** In general, $f \circ g \neq g \circ f$. Composition is not commutative.
- **Finding the Inverse (f^{-1}):**
  1. First, prove that the function is a bijection. If it isn't, it's not invertible.
  2. Write the function as $y = f(x)$.
  3. Swap the variables x and y.
  4. Solve the new equation for y. This expression for y is $f^{-1}(x)$.

✍️ Worked Example

Question: Consider the function $f: R_+ \to [4, \infty)$ given by $f(x) = x^2 + 4$. Show that f is invertible and find the inverse of f.

Solution:

1. **Check for Bijection:**
   - **One-one:** Let $x_1, x_2 \in R_+$ (positive real numbers). Assume $f(x_1) = f(x_2)$. $x_1^2 + 4 = x_2^2 + 4 \implies x_1^2 = x_2^2$. Since the domain is R_+, we take only the positive square root, so $x_1 = x_2$. Hence, f is **one-one**.
   - **Onto**: Let $y \in [4, \infty)$ (the codomain). Set $y = f(x)$. $y = x^2 + 4 \implies x^2 = y - 4 \implies x = \sqrt{y-4}$. Since $y \geq 4$, $y - 4 \geq 0$, so $\sqrt{y-4}$ is a real number. Also, the square root symbol implies the positive root, so $x \in R_+$. Thus, for any y in the codomain, we found a valid x in the domain. Hence, f is **onto**.
   - Since f is a bijection, it is **invertible**.
2. **Find the Inverse:**
   - Step 1: Write as $y = x^2 + 4$.

- Step 2: Swap x and y: $x = y^2 + 4$.
- Step 3: Solve for y: $y^2 = x - 4 \implies y = \sqrt{x-4}$.
- Thus, $f^{-1}(x) = \sqrt{x-4}$.

⚠️ **Pitfall Analysis**

- **Mistake**: Confusing the order of composition. Students often calculate $f(g(x))$ when asked for $(g \circ f)(x)$. Remember, $(g \circ f)(x)$ means $g(f(x))$. The function on the right acts first.
- **Mistake**: Trying to find the inverse of a function without first proving it is bijective. This is a fundamental conceptual error. If a function isn't one-one, its inverse would have one input mapping to multiple outputs, which violates the definition of a function. If it isn't onto, there would be elements in the domain of the inverse that have no image.

Binary Operations

This concept generalizes operations like addition and multiplication.

🧠 **Memory Boost: Quick Refresher**

- A **function** from $A \times A$ to A takes an ordered pair of elements from set A and gives back a single element, also from set A.

📚 A **binary operation** `*` on a set A is a function `*`: $A \times A \to A$. We denote `*` (a, b) by $a * b$.

- **Simple Explanation**: It's a rule for combining any two elements of a set to produce a third element that is *also in the same set*.
- **Example**: Addition `+` on the set of natural numbers N. If you take any two natural numbers a, b, their sum $a + b$ is also a natural number. So, addition is a binary operation on N.

- **Counter-Example**: Subtraction `-` is *not* a binary operation on N because for $a = 3, b = 5$, $a - b = -2$, which is not in N.

Properties of Binary Operations

Property	Condition	Example (with + on Z)
Commutative	$a * b = b * a$ for all $a, b \in A$.	$3 + 5 = 5 + 3$
Associative	$(a * b) * c = a * (b * c)$ for all $a, b, c \in A$.	$(2 + 3) + 4 = 2 + (3 + 4)$
Identity Element	There exists an element $e \in A$ such that $a * e = e * a = a$ for all $a \in A$.	For addition on Z, the identity is 0, since $a + 0 = a$.
Inverse of an Element	For an element $a \in A$, there exists an element $b \in A$ such that $a * b = b * a = e$, where e is the identity element. b is the inverse of a.	For addition on Z, the inverse of a is $-a$, since $a + (-a) = 0$.

✍️ Worked Example

Question: Determine whether the operation `*` on Q (set of rational numbers) defined by $a * b = ab + 1$ is commutative and associative.

Solution:

1. **Commutativity:**
   - $a * b = ab + 1$
   - $b * a = ba + 1$
   - Since multiplication of rational numbers is commutative ($ab = ba$), we have $ab + 1 = ba + 1$.
   - Therefore, $a * b = b * a$. The operation is **commutative.**
2. **Associativity:**
   - $(a * b) * c = (ab + 1) * c = (ab + 1)c + 1 = abc + c + 1.$
   - $a * (b * c) = a * (bc + 1) = a(bc + 1) + 1 = abc + a + 1.$
   - Since $abc + c + 1 \neq abc + a + 1$ (in general), the operation is **not associative.**

⚠️ Pitfall Analysis

- **Mistake**: Assuming an operation is associative just because it's commutative. As the example above shows, these are independent properties. Always check them separately.
- **Mistake**: Incorrectly identifying the identity element. For multiplication `*` on Q, the identity is 1. But for the operation `*` defined as $a * b = a + b - 1$, the identity element e would be found by solving $a * e = a \implies a + e - 1 = a \implies e = 1$.

🎯 Exam Corner & Chapter Summary

- **Weightage:** The unit 'Relations and Functions' typically carries around 8 marks.
- **Most Important Topics**:
  1. **Equivalence Relations:** Proving a relation is reflexive, symmetric, and transitive is a very common 3 or 4-mark question.
  2. **Bijective Functions:** Proving a function is one-one and onto is another standard high-marks question.
  3. **Invertible Functions**: Finding the inverse of a function is frequently asked, often combined with the proof of it being a bijection.
- **Question Types:**
  - Short Answer (1-2 marks): Check if a given relation is symmetric, or if a function is one-one.
  - Long Answer (3-4 marks): Show that a relation is an equivalence relation. Show that a function is bijective and find its inverse.
- **Final Advice**: This chapter is more about logic and proof than heavy calculation. Write your proofs step-by-step, clearly stating the definitions of reflexive, symmetric, transitive, one-one, and onto as you use them. Clarity in your answers will fetch you full marks. them. Clarity in your answers will fetch you full marks.